

# On the completeness of trajectories for some Mechanical systems

Miguel Sánchez Caja

**Abstract** The classical tools which ensure the completeness of vector fields and second order differential equations for mechanical systems are revisited. Possible extensions are discussed in three directions: infinite dimensional Banach and Hilbert manifolds, Finsler metrics and pseudo-Riemannian spaces, including links with some relativistic spacetimes. Special emphasis is taken in the cleaning of known techniques, the statement of open questions and the exploration of prospective frameworks.

## 1 Introduction

As explained in classical Abraham & Marsden book [1, p. 71], the completeness of vector fields *is often stressed in the literature since it corresponds to well-defined dynamics persisting eternally*. However, in many circumstances one has to live with incompleteness and, in this case, incompleteness may correspond with the failure of our model. Remarkably, this happens in General Relativity, where singularities have become so common (Schwarzschild spacetime, Raychauduri equation, theorems by Penrose and Hawking...) that one expects to find incompleteness under general physically reasonable assumptions —and hope that the quantum viewpoint will be able to explain what such a singularity will mean. At any case, to determine the completeness or incompleteness of the system becomes a fundamental property.

In his early works at the beginning the the seventies, Marsden gave some two remarkable results on completeness. The first one, in collaboration with Weinstein [62], on the completeness of Hamiltonian vector fields, extends previous works by

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Gordon [29], Ebin [22] and others. The second one, on the completeness of compact homogeneous pseudo-Riemannian manifolds [41], one of the scarce results ensuring completeness instead of incompleteness in the Lorentzian setting at that epoch. The results on the side of geodesics in the Lorentzian setting have increased notably since then (see the review [17]) and, in fact, some connections with the original Riemannian results for Hamiltonian systems have appeared. This has been an stimulus for a recent update of the classical Riemannian results carried out by the author and his coworkers [14].

The aim of the present paper is to revisit these results, formulating them in a general framework, and pointing out new open questions and lines of study. The paper is organized in three parts. In the first one (Section 2), some preliminaries on infinite-dimensional Banach manifolds endowed with Finsler metrics are introduced. The reason is that, under our viewpoint, this is the general natural framework for the completeness of vector fields (first order problems) and, eventually, some second order problems might be extended to this setting.

In Section 3 we study completeness for both, first and second order equations. For first order, we review some old results [29, 22, 62, 1, 2] formulating them in the general Banach Finsler case, and allowing also the time-dependence of the vector fields. The introduced *primary bounds* (Definition 1) allow to purify techniques, Theorem 1. For second order, i.e., trajectories accelerated by potentials and other time-dependent forces, we give a general result on completeness in Riemannian Hilbert manifolds (Theorem 2), which summarizes and extends those in [29, 22, 62, 14]. The latter are also simplified technically because, even though our proof uses comparison criteria between differential equations as in previous references, here such criteria are reduced essentially to the elementary Lemma 1 —and the bounds through positively complete functions introduced in [62] reduce to primary bounds too. We suggest the possibility to go further in two directions: the time-dependence of the potentials and the Finsler Banach framework.

In Section 4 we focus on (finite-dimensional) pseudo-Riemannian manifolds. Here there is a big diversity of results and techniques (see [17]) and we concentrate in two topics. First, results in manifolds with a high degree of symmetry. In particular, the extension of Marsden's Theorem 5 to conformal metrics (Theorem 6) is explained by taking into account techniques in previous section. Second, the geometry of wave type spacetimes, which provides a link between Riemannian and Lorentzian results (Theorem 9) with new exciting open questions —some of them collected at the end.

## 2 Preliminaries on infinite-dimensional manifolds

Some preliminaries on Banach manifolds are introduced, gathering results on the elements which will be relevant for the posterior results, and the framework for tentative generalizations. Special emphasis is put in the necessity of paracompactness for the ambient manifold, as it will be equivalent to the existence of a  $C^0$ -Finsler

metric such that its associated distance metrizes the manifold topology. The role of smoothability for Finsler metrics is also emphasized because, essentially,  $C^0$  suffices for distance estimates in first order problems (Section 3.1), but further smoothability may be required for the development of the second order (Section 3.2).

We will follow conventions on Banach and Hilbert manifolds as in the original papers by Palais [47, 46, 45], as well as books such as Abraham, Marsden & Ratiu [2], Lang [40], Deimling [19], Kriegl & Michor [38] or Moore's notes [43].

**Topological conventions on Banach manifolds.** Banach manifolds will be always assumed  $C^k$  with  $k \geq 1$ , as well as *connected, Hausdorff and paracompact* and, thus, normal<sup>1</sup>. A  $n$ -manifold will be a finite dimensional Banach manifold with dimension  $n \in \mathbb{N}$ . The letter  $M$  will denote a manifold; when the infinite dimension is allowed, we will remark explicitly that  $M$  is Banach (say, modelled on some Banach space  $B$  with norm  $\|\cdot\|$ ) or, when applicable, Hilbert (modelled on some real Hilbert space  $H$  with inner product  $\langle \cdot, \cdot \rangle$ ). When indefinite metrics are considered, as in Section 4,  $M$  will be typically a  $n$ -manifold.

**Finsler Banach manifolds.**  $F$  will denote a (reversible) Finsler metric on the Banach manifold  $M$ , and  $(M, F)$  will be called a Finsler Banach manifold. This notion is taken in the sense of Palais [46], that is,  $F$  yields a norm at each tangent space:

$$F_p : T_p M \rightarrow \mathbb{R} \quad (1)$$

which admits a  $C^k$  chart  $(U, \phi)$ ,  $p \in U$ ,  $\phi : U \subset M \rightarrow B$  such that the induced norms

$$\|u\|_q := F_q(d(\phi^{-1})_{\phi(q)}(u)) \quad \forall u \in B, \quad (2)$$

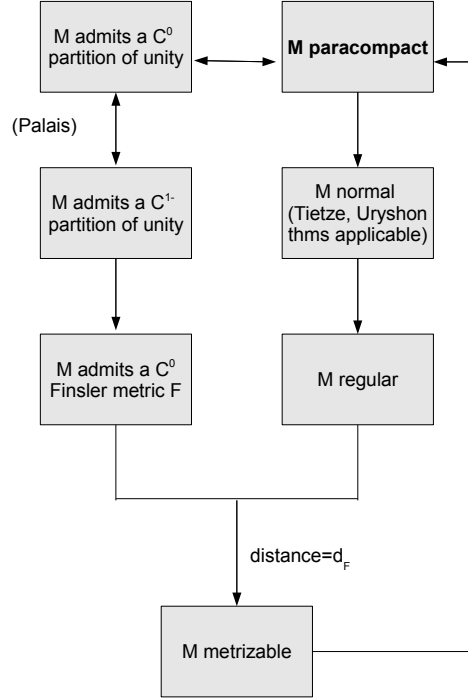
(where  $d$  denotes here the differential or tangent map) satisfy: (a) they are equivalent to the natural norm  $\|\cdot\|$  of  $B$  (i.e.,  $\varepsilon_q \| \cdot \|_q \leq \| \cdot \| \leq \varepsilon_q^{-1} \| \cdot \|_q$  for some  $0 < \varepsilon_q < 1$  and all  $q \in U$ ), and (b) they vary continuously at  $p$  (i.e., for each  $0 < \varepsilon < 1$  there exists a neighborhood  $U_\varepsilon \subset U$  of  $p$  such that  $\varepsilon \| \cdot \|_q \leq \| \cdot \|_p \leq \varepsilon^{-1} \| \cdot \|_q$  for all  $q \in U_\varepsilon$ ).

As norms cannot be differentiable at<sup>2</sup> 0, the  $C^{k'}$  differentiability of the norm  $\|\cdot\|$  means always away from 0. The Finsler metric is called  $C^{k'}$  (for  $0 \leq k' \leq k-1$ ) if  $F_p$  is  $C^{k'}$  and varies smoothly with  $p$  in a  $C^{k'}$  way (i.e., for any chart  $(U, \phi)$  as above the map  $U \times (B \setminus \{0\}) \rightarrow \mathbb{R}$ ,  $(q, u) \mapsto \|u\|_q$  is  $C^{k'}$ ).

**Existence of Finsler metrics.** The question of the existence of a  $C^0$  Finsler metric depends only on topological grounds, but the existence of a  $C^{k'}$  one with  $k' > 0$  is much subtler. Namely, on the one hand the hypothesis of paracompactness on  $M$

<sup>1</sup> In particular, our Banach manifolds will be always regular and, so, some difficulties pointed out by Palais in [47] (see Sect. 2 including the Appendix therein), will not apply. The central role of paracompactness from the topological viewpoint is stressed in Figure 1. Notice that, as a difference with the finite dimensional case, second countability does not imply paracompactness (see for example [42], [38, Sect. 27.6] or [47]).

<sup>2</sup> Recall that neither the absolute value is. Moreover, at least in the finite-dimensional case, the square of a norm is smooth at 0 if and only if it comes from a scalar product [61, Prop. 4.1].

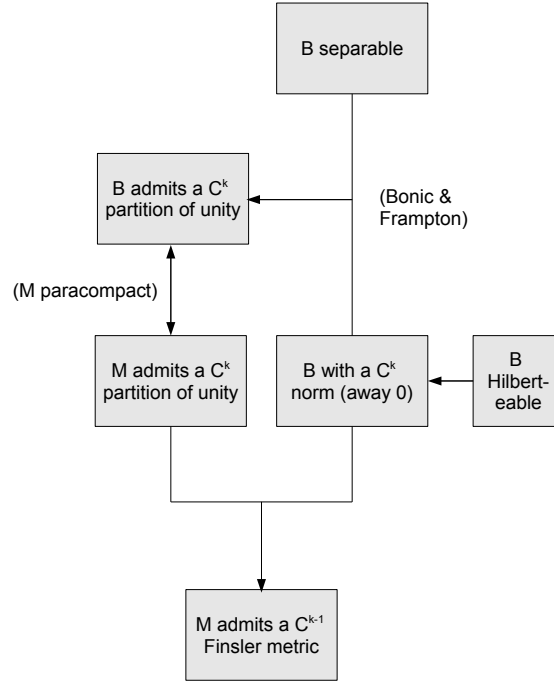


**Fig. 1** Topological properties related to the paracompactness of a (connected, Hausdorff) Banach manifold.

becomes equivalent to the existence of  $C^0$ -partitions of the unity subordinated to any open covering and, by a result of Palais [46, Th. 1.6], [47, Sect. 3], it is also equivalent to the existence of locally Lipschitz partitions of the unity. The latter allows to ensure the existence of  $C^0$  Finsler metrics in any Banach manifold [46, Th. 2.11]. On the other hand, when the model Banach space  $B$  admits  $C^k$  partitions of the unity subordinate to any open covering (which happens, in particular, when  $B$  is separable and admits a  $C^k$  norm away from 0, see [8], [2, Prop. 5.5.18, 5.5.19]), then the Banach manifold  $M$  also admits  $C^k$  partitions of the unity [2, Th. 5.5.12] and, in this case,  $M$  admits  $C^{k-1}$ -Finsler metrics too (Figure 2).

*Remark 1.* It is worth pointing out that, even though the differentiability of  $F$  may be useful for some issues (see Section 3.2.2 below), it will not be especially relevant for the estimates which involve length or distances in the problems of complete-

ness of trajectories, to be studied in Section 3.1. This is interesting for the time-dependent problem, as this case is commonly handled by transforming it in a non time-dependent one on a product manifold  $M \times \mathbb{R}$ , endowed with the direct sum Finsler metric (obtained by adding the Finsler metrics of the factors), see Remark 3. Nevertheless, this direct sum is non-differentiable away from 0 even if the metric on each factor is (the differentiability on vectors of the product with one of the two components equal to zero is not guaranteed).



**Fig. 2** Existence of smooth Finsler metrics on the manifold  $M$  modelled on the Banach space  $B$ .

**Associated distance.** Recall that our definition of Finsler metric  $F$  includes *reversibility* (i.e.,  $F(v) = F(-v)$  for all tangent vector  $v \in TM$ ) and, so,  $F$  defines a natural distance by taking the infimum of the lengths of the curves connecting each pair of points. This distance will be denoted  $d_F$  or, simply,  $d$  if there is no possibility of confusion. One can prove that the topology generated by  $d$  agrees with the

manifold topology by using the regularity of the manifold. [47, p. 202] and, so, that all Finsler Banach manifolds are metrizable<sup>3</sup>.

We will speak on the completeness of  $(M, F)$  in the sense of the convergence of Cauchy sequences for  $(M, d)$ . One can also consider geodesics for  $(M, F)$  (as locally length-minimizing curves of constant speed, with other characterizations under further smoothability, see Section 3.2.2) and we will say that  $(M, F)$  is geodesically complete when its inextensible geodesics are defined on all  $\mathbb{R}$ . Clearly, the existence of an incomplete geodesic implies the incompleteness of  $d$ .

In order to make estimates with the distances, we fix a *base* point  $p_0 \in M$  and denote

$$|p| = d(p, p_0) \quad \forall p \in M. \quad (3)$$

(This notation will be used when the properties under study are independent of the chosen point  $p_0$ ).

**Pseudo-Riemannian metrics on Banach manifolds.** When the model space  $B$  of the Banach manifold  $M$  is reflexive, it is natural to define a  $C^{k'}$  ( $k' \leq k-1$ ) pseudo-Riemannian metric  $g$  as a  $C^{k'}$  choice of a continuous symmetric bilinear form  $g_p$  at each tangent space  $T_p M$  such that the associated *flat* map (to lower indexes in finite dimension) into the dual space given by

$$\flat_p : T_p M \rightarrow T_p M^*, \quad v_p \mapsto g_p(v_p, \cdot) \quad (4)$$

is a homeomorphism (if this condition on  $\flat_b$  were not imposed, one would speak of a *weak* pseudo-Riemannian metric and the reflexivity of  $B$  would not be required). The set of all such bilinear forms  $g_p$  can be identified via a chart around  $p$  with an open subset of the set  $\text{BL}_{\text{sym}}(B)$  of all the continuous symmetric bilinear forms on  $B$ . As  $\text{BL}_{\text{sym}}(B)$  is naturally a Banach space too, the pseudo-Riemannian metric  $g$  can be regarded as a section of a fiber bundle on  $M$  with fiber  $\text{BL}_{\text{sym}}(B)$  (see [40, Ch VII.1]).

**Riemannian metrics on Hilbert manifolds.** When the pseudo-Riemannian metric  $g$  is positive definite then we say that it is Riemannian. As we are assuming that  $\flat_p$  is a homeomorphism, the model space  $B$  is Hilbertizable. So, it will be denoted  $H$ , and we will consider only Riemannian metrics on Hilbert manifolds. Notice that, for any Riemannian metric  $g$  one has an associated Finsler metric given as  $F(v) = \sqrt{g(v, v)}$  for all  $v \in TM$ . So, the bounds required in the definition of continuity for  $F$  in the Finslerian case (see (a) and (b) below formula (2)), hold here in terms of the norm associated to the inner product  $\langle \cdot, \cdot \rangle$  of  $H$ . Moreover, this norm is always  $C^\infty$  away from 0, any  $C^k$  Hilbert manifold modelled on a separable space  $H$  admits  $C^k$  partitions of the unity and, then, a  $C^{k-1}$  Riemannian metric. Riemannian metrics on Hilbert manifolds, as well as their geodesics, are extensively studied in the literature, see for example [40] or, for the separable case, [37]. The Hopf-Rinow theorem for

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<sup>3</sup> Consistently, paracompactness can be deduced from the hypothesis of metrizability (or even just from pseudo-metrizability, see [2, Lemma 5.515]).

separable Riemann Hilbert manifolds can be found in [37, Th. 2.1.3] (including the “Notes” therein).

**Concluding remarks and conventions.** For the convenience of the reader, a summary on the topological/smooth results commented above is provided in Figures 1 and 2. Basic detailed background can be found in [46, 47] and [2]. In what follows, all the objects will be *smooth* i.e. as differentiable as possible according to the discussion above. This will mean at least  $C^2$  for any Banach manifold  $M$  and  $C^1$  for any vector field  $X$  on  $M$ , in the case of first order problems (Section 3.1); as emphasized in Remarks 1 and 3, Finsler metrics are required only  $C^0$  at this stage. Further requirements of smoothability will be needed for the second order case (Section 3.2). In the (indefinite) finite-dimensional case (Section 4), the issues on smoothability are not specially relevant and, so, the reader may either track them or just assume  $C^\infty$  smoothability.

### 3 Completeness of trajectories in a positive-definite infinite-dimensional setting

This section is divided into two subsections. The first one is devoted to the (first order) problem of the completeness of a vector field. We review some results essentially known from the seventies [29, 22, 62] and explained in the books [1, 2], extending them to the ( $C^0$ ) Finsler setting when possible (Propositions 1, 2). The notion of *primarily complete* function and, then, primary bounds for a vector field (Definition 1) allows to give an optimal result in the Finsler Banach case, Theorem 1. The time-dependent case is specially discussed in Remark 3 and the last part of the subsection.

In the second subsection, Theorem 2 (plus Remark 5) summarizes and extends the results on second order differential equations in [29, 22, 62, 14]. The proof is carried out in three conceptually independent steps, and the systematic usage of simple Lemma 1 allows to carry out easily the technical bounds of the second step. The relation between the previous notion of *primarily complete* function and Abraham-Marsden’s one of *positive completeness* is discussed. Even though the obtained bounds for the potential in the time-dependent case are natural, some alternative are discussed suggesting the possibility of further results. We also discuss the difficulties of the generalization when Riemannian metrics are replaced by Finslerian ones, including a simple example of result for the (standard) finite-dimensional Finsler case.

### 3.1 Complete vector fields on Finsler Banach manifolds

#### 3.1.1 Elementary criteria

The properties of the (local) flow  $\phi$  of a vector field  $X$  and, in particular, the existence of a flow box around each point, can be found, for example, in [2, p. 192ff], [40, p. 84ff] or [43, Sect. 1.10]. We start with a well-known result (see for example [2, Prop. 4.1.19]).

**Proposition 1.** *Let  $X$  be a vector field on a Banach manifold  $M$ , and let  $c : [0, b) \rightarrow M$  (resp.  $[-b, 0) \rightarrow M$ ) be an integral curve of  $X$  with  $0 < b < +\infty$ . Then,  $c$  can be extended beyond  $b$  as an integral curve of  $X$  if and only if there exists a sequence  $t_n \rightarrow b^-$  such that the sequence  $\{c(t_n)\}_n$  (resp.  $\{c(-t_n)\}_n$ ) is convergent in  $M$ .*

*Proof.* The necessity of the condition is obvious. For its sufficiency, let  $p \in M$  be the limit of the sequence. The existence of a flow box of  $X$  at  $p$  ensures the existence of a neighborhood  $U$  of  $p$  and some  $\varepsilon > 0$  such that the integral curves of  $X$  at any  $p' \in U$  are defined on  $(-\varepsilon, \varepsilon)$ . So, taking  $n$  large so that  $b - t_n < \varepsilon$  the integral curve through  $c(t_n)$  will be defined on  $[0, \varepsilon)$  and  $c$  will be extensible through  $b$ .

Accordingly, we will say that an integral curve  $c$  of  $X$  defined on some interval  $I$  of  $\mathbb{R}$  is *complete* if it can be extended as an integral curve of  $X$  to all  $\mathbb{R}$ , and  $X$  will be complete if so are its integral curves.

*Remark 2.* This result follows in the infinite-dimensional case as well as in the finite-dimensional one. However, the application in the latter case is easier, because  $M$  is then locally compact. For example, Proposition 1 yields directly that, if the support of  $X$  is compact (in particular, if  $M$  is compact and, thus, finite-dimensional) then  $X$  is complete.

Analogously, one can prove that if a Banach manifold  $(M, F)$  admits a  $C^1$ -proper map  $f : M \rightarrow \mathbb{R}$  (i.e.  $f^{-1}([a, b])$  is compact for any compact  $[a, b] \subset \mathbb{R}$ ), then a vector field  $X$  is complete whenever

$$|X_p(f)| \leq C_1 |f(p)| + C_2 \quad (5)$$

for some  $C_1, C_2 > 0$  and all  $p \in M$  (recall that (5) implies a bound for the derivative of  $\log(C_1 |f \circ c| + C_2)$  and, if the domain of the integral curve  $c$  is bounded, also a bound for  $f$  on  $c$ , which yields the result from Proposition 1, as it can be applied because  $f$  is proper, see [1, 2.1.20] or [2, 4.1.21] for more details). Even though proper maps are well behaved in Banach manifolds (for example, they are closed maps [48]) results as the previous one are used typically in the finite-dimensional case, putting, say,  $f = C_1 |x|^2 + C_2$  on a complete Riemannian  $n$ -manifold.

The following criterion on completeness for Finsler Banach manifolds holds as in the case of Riemann Hilbert ones or Banach spaces (compare with [1, Prop. 2.1.2] or [2, Prop. 4.1.22]).



**Proposition 2.** *Let  $(M, F)$  be a complete Finsler Banach manifold and  $X$  a vector field on  $M$ . If  $c : I \subset \mathbb{R} \rightarrow M$  is a integral curve of  $X$  and  $F(\dot{c})$  is bounded on bounded subintervals of  $I$ , then  $c$  is complete.*

*Proof.* Assume with no loss of generality that  $I = [0, b)$ ,  $b < \infty$ , let  $A$  be the assumed bound and choose  $\{t_n\} \nearrow b$ . The associated distance  $d$  satisfies then:

$$d(c(t_n), c(t_m)) \leq \int_{t_n}^{t_m} F(\dot{c}(t)) dt \leq A|t_n - t_m|.$$

So,  $\{c(t_n)\}_n$  is a Cauchy sequence, which becomes convergent to some limit  $p$  by the completeness of  $(M, F)$ . Then, Proposition 1 can be applied to  $\{c(t_n)\}_n$ .

*Remark 3. (The time-dependent case.)* The results in the previous two propositions can be extended to the case when  $X$  is time-dependent, and defined for all the values of the time.

More precisely, consider the product manifold  $M \times \mathbb{R}$ , let  $\Pi_{\mathbb{R}} : M \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\Pi_M : M \times \mathbb{R} \rightarrow M$  be the natural projections, and denote by  $t$  the natural coordinate on  $\mathbb{R}$ . We say that  $X$  is a *time-dependent vector field on  $M$*  if it is a smooth section of the pull-back bundle  $\Pi_M^*(TM)$ , so that its base is  $M \times \mathbb{R}$  and each fiber comes from a tangent space to  $M$ . Such a vector field yields naturally a (time-independent) vector field  $\hat{X}$  on  $M \times \mathbb{R}$  which satisfies  $d\Pi_M \hat{X}_{(p_0, t_0)} = d\Pi_M X_{(p_0, t_0)}$  and  $d\Pi_{\mathbb{R}} \hat{X}_{(p_0, t_0)} = \partial_t|_{t_0}$  for all  $(p_0, t_0) \in M \times \mathbb{R}$ .

To speak on the integral curves of  $X$  makes a natural sense (see for example [40, Ch. IV]) and becomes equivalent to consider the integral curves of  $\hat{X}$ ; in fact,  $c$  will be an integral curve of  $X$  if and only if  $\hat{c} : t \mapsto (c(t), t)$  is an integral curve of  $\hat{X}$ . So, Proposition 1 is extended directly to a time-dependent  $X$ .

For Proposition 2, recall that, if  $(M, F)$  is a Finsler Banach manifold, then  $M \times \mathbb{R}$  admits a natural  $C^0$  Finsler metric  $\hat{F}$  obtained as the direct sum of  $F$  and the usual one on  $\mathbb{R}$  (see Remark 1). Clearly,  $\hat{F}$  will be complete if and only if so is  $F$  and the  $F$ -length of the integral curve  $c$  of  $X$  is bounded on finite intervals if and only so is the  $\hat{F}$ -length of the integral curve  $\hat{c}$  of  $\hat{X}$ , as required.

### 3.1.2 Applications

Next, we apply previous results to simple but general situations. Previously, we consider a technical elementary result for future referencing.

**Lemma 1.** *Consider the equation*

$$\dot{u} = f(t, u) \quad \text{on} \quad [t_0, T), \quad (6)$$

where  $f \in C^0(\mathbb{R}^2, \mathbb{R})$  is locally Lipschitz in its second variable, and let  $w = w(t)$  be a subsolution of the differential equation i.e.,  $\dot{w} < f(t, w)$  on  $[t_0, T)$ . Then for every solution  $u = u(t)$  of (6) such that  $w(t_0) \leq u(t_0)$  we have

$$w(t) < u(t) \quad \text{for all} \quad t \in (t_0, T).$$

The proof follows just recalling that  $\Delta := w - u < 0$  close to  $t_0$  by the assumptions and, if there were a first point such that  $\Delta(t_1) = 0$ , then  $\dot{\Delta}(t_1) < 0$ , a contradiction (see for example [60, Lemma 1.1])

**Estimates of the growth for completeness.** Let us introduce some auxiliary definitions.

**Definition 1.** A (locally Lipschitz) function  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  is *primarily complete* if it is positive, non-decreasing and satisfies:

$$\int_0^\infty \frac{dx}{\alpha(x)} = \infty \quad (7)$$

A vector field  $X$  on a Finsler Banach manifold  $(M, F)$  is *primarily bounded* if there exists a primarily complete function  $\alpha$  (the *bounding function*) such that

$$F(X_p) < \alpha(|p|) \quad \forall p \in M. \quad (8)$$

In particular,  $X$  *grows at most linearly* if it is primarily bounded by an affine bounding function, i.e.:

$$F(X_p) < C_0 + C_1|p| \quad \forall p \in M, \quad (9)$$

for some constants  $C_0, C_1 > 0$ .

*Remark 4.* The best polynomial candidate for a bounding function  $\alpha$  has degree one as, clearly, no polynomial of higher degree can be a primarily complete function. Nevertheless, a slightly faster growth is allowed for non-polynomial functions. For example,  $\alpha$  will be primarily complete if it grows as  $1 + x \cdot \log x \cdot \log(\log x)$  for large  $x$  (see also the discussion in the last part of Section 3.2.1).

Now, we can give a general bound for the completeness of vector fields.

**Theorem 1.** Any primarily bounded vector field on a complete Finsler Banach manifold  $(M, F)$  is complete.

*Proof.* Let  $c : I \rightarrow M$  be an integral curve of  $X$ . With no loss of generality, assume  $I = [0, b)$ , and choose  $p_0 = c(0)$  in the notation introduced in (3). Then:

$$|c(t)| \leq \int_0^t F(\dot{c}(s)) ds \leq \int_0^t \alpha(|c(s)|) ds. \quad (10)$$

where  $\alpha$  is the bounding function. Thus, putting  $w(t) = |c(t)|$ :

$$\dot{w}(t) < \alpha(w(t)).$$

The unique inextensible solution  $w_0$  of the equality

$$\dot{w}_0(t) = \alpha(w_0(t)) \quad w_0(0) = w(0)(= 0)$$

is defined for all  $t \in [0, \infty)$ , as its inverse is determined as  $w \mapsto t(w) = \int_0^w d\bar{w} / \alpha(\bar{w})$  and (7) holds. So, from Lemma 1 one has

$$w(t) < w_0(t) < w_0(b) \quad \forall t \in (0, b).$$

As  $\alpha$  is non-decreasing, equation (8) yields the bound  $F(\dot{c}) \leq \alpha(w_0(b))$  so that Proposition 2 is applicable.

By considering on  $\mathbb{R}$  a vector field type  $X_{x_0} = \alpha(x_0)\partial_x$  one can check the optimality of Theorem 1 and, in particular, the optimality (as discussed in Remark 4) of the at most linear growth of  $X$  to ensure completeness. Of course, a vector field with a *superlinear* growth such as  $X = y^2\partial_x$  may be complete. In fact, in order to ensure completeness, only the growth of  $X$  along the direction of its integral curves becomes relevant. This underlies in the fact that the sum of two complete vector fields  $X, Y$  may be incomplete (put  $Y = x^2\partial_y$  and  $X$  as before) and may suggest more refined hypotheses for completeness in Hilbert spaces (compare with [2, Exercise 2.2H]).

**Time-dependent case.** As in the case of the criterions on completeness, Theorem 1 can be extended to the case of a time-dependent vector field  $X$ . In fact, the proof works in a completely analogous way (with the observations in Remark 3), if the inequality in (7) is regarded as  $F(X_{(p,t)}) < \alpha(|p|)$  for all  $(p, t) \in M \times \mathbb{R}$ . Nevertheless, one can be a bit more accurate.

**Definition 2.** A time-dependent vector field  $X$  on a Finsler Banach manifold is *primarily bounded along finite times* if there exists a primarily complete function  $\alpha$  and a continuous function  $C(t) > 0$  such that

$$F(X_{(p,t)}) < C(t)\alpha(|p|) \quad \forall (p, t) \in M \times \mathbb{R}.$$

In particular,  $X$  *grows at most linearly along finite times* when  $\alpha$  can be chosen affine or, equivalently, when

$$F(X_{(p,t)}) < C_0(t) + C_1(t)|p| \quad \forall (p, t) \in M \times \mathbb{R} \quad (11)$$

for some functions  $C_0(t), C_1(t) > 0$

**Corollary 1.** *Let  $X$  be a time-dependent vector field on a complete Finsler Banach  $(M, F)$ . If  $X$  is primarily bounded along finite times then it is complete.*

*Proof.* Reasoning with an integral curve  $c$  defined on  $[0, b)$  as in the proof of Proposition 2, notice that the inequality (11) for all the pairs  $(p, t) \in M \times [0, b]$  also yields a time independent inequality as (9) with  $C_i = \text{Max}_{t \in [0, b]} \{C_i(t)\}, i = 0, 1$ . Then, reason as in Remark 3 taking into account that  $\hat{X}$  is primarily bounded (on  $M \times [0, b]$ ) if and only if so does  $X$ .

### 3.2 Completeness for 2nd order trajectories

#### 3.2.1 General result on Riemann Hilbert manifolds

The next result, stated on a Riemann Hilbert manifold  $(M, g)$ , will summarize those in [29, 22, 1, 14]. To state it, notice that the notion of time-dependent vector field on  $M$  in Remark 3 can be directly translated to (continuous, linear) endomorphism fields, which will be then regarded as sections on  $M \times \mathbb{R}$  with fiber at each  $(p, t) \in M \times \mathbb{R}$  equal to the vector space of bounded linear operators  $T_{(p,t)}(M \times \mathbb{R}) \rightarrow T_{(p,t)}(M \times \mathbb{R})$  which vanish on  $(0, \partial_t)_{(p,t)}$ . Given such a field  $E$ , we will decompose it as  $E = S + H$  where  $S$  denotes its self-adjoint part ( $S = (E + E^\dagger)/2$ ), and  $H$  the skew-adjoint one. A time-dependent or non-autonomous potential means just a smooth map  $V : M \times \mathbb{R} \rightarrow \mathbb{R}$ , then, the notation  $\partial V / \partial t : M \times \mathbb{R} \rightarrow \mathbb{R}$  makes a natural sense, and  $\nabla^M V$  denotes the time dependent vector field on  $M$  obtained by taking the gradient of  $V$  at each slice  $t = \text{constant}$  with respect to  $g$ , i.e.,  $dV(X(p, t), 0) = g_p(\nabla^M V(p, t), X(p, t))$  for  $(X(p, t), 0) \in T_{(p,t)}(M \times \mathbb{R})$ . The pointwise norm induced by  $g$  in any space of tensor fields will be denoted  $\|\cdot\|$ .

**Theorem 2.** *Let  $(M, g)$  be a complete Riemann Hilbert manifold, and consider a time-dependent endomorphism field  $E = S + H$ , a vector field  $R$  and a potential  $V$  on  $M$ , all of them time-dependent and smooth. Assume that:*

- (i)  *$S$  is uniformly bounded along finite times, i.e.,  $\|S_{(p,t)}\| \leq C_0(t)$  for all  $(p, t) \in M \times \mathbb{R}$ ,*
- (ii)  *$R$  grows at most linearly along finite times, i.e.,  $\|R_{(p,t)}\| \leq C_0(t) + C_1(t)|p|$  for all  $(p, t) \in M \times \mathbb{R}$ , and*
- (iii) *both,  $-V$  and  $|\partial V / \partial t|$  grow at most quadratically along finite times, i.e., they are bounded by  $C_0(t) + C_2(t)|p|^2$ ,*

where  $C_i(t), i = 0, 1, 2$ , denote positive functions. Then, the inextensible solutions of

$$\frac{D\dot{\gamma}}{dt}(t) = E_{(\gamma(t), t)} \dot{\gamma}(t) + R_{(\gamma(t), t)} - \nabla^M V(\gamma(t), t), \quad (12)$$

are complete.

*Proof.* In order to clarify the ideas, the proof is divided into three steps.

*Step 1: Reduce the problem to the completeness of a vector field on the tangent bundle.* The second order equation (12) allows to define a vector field  $G$  on the manifold  $T(M \times \mathbb{R})$  such that each solution  $\gamma$  of (12) generates an integral curve  $t \mapsto (\gamma(t), 1)$  of  $G$ . This is standard (see for example [1, Ch. 3] or, for explicit details on the time-dependent case, [14, Section 3.1]) and, so, the problem will be reduced to apply the criterions in Propositions 1 and 2 to  $G$ .

*Step 2: Bound the velocity of any solution  $\gamma$  of (12), by using the hypotheses (i) to (iii).* With no loss of generality, let  $\gamma : [0, b) \rightarrow M, b < \infty$  be a solution of (12) whose extendability to  $b$  is to be determined, let  $u(t) = g(\dot{\gamma}(t), \dot{\gamma}(t))$  the function to

be bounded, and choose the base point  $p_0 = \gamma(0)$  for (3). Taking in (12) the product by  $\dot{\gamma}$ :

$$\frac{1}{2}\dot{u}(t) = g(S_{(\gamma(t),t)}\dot{\gamma}(t), \dot{\gamma}(t)) + g(R_{(\gamma(t),t)}\dot{\gamma}(t)) - \left( \frac{d}{dt}V(\gamma(t),t) - \frac{\partial V}{\partial t}(\gamma(t),t) \right)$$

so that taking pointwise norms and simplifying the notation:

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}u + V) &\leq \|S\| u + \|R\| \sqrt{u} + \partial V / \partial t \\ &\leq (\|S\| + 1/2)u + \|R\|^2 / 2 + \partial V / \partial t \end{aligned} \quad (13)$$

Using the bounds (i), (ii), (iii) and taking into account that, as the  $t$  coordinate is confined in the compact interval  $[0, b]$ , the  $t$ -dependence of these bounds can be dropped:

$$\frac{d}{dt}(u + 2V) \leq A_0 + A_1 u + A_2 |\gamma|^2 \quad (14)$$

for some constants  $A_0, A_1, A_2 > 0$ . Consider the function  $l(t) = \int_0^t \sqrt{u}$ ,  $t \in [0, b]$  which provides the length of  $\gamma$ . Clearly:

$$|\gamma(t)|^2 \leq l(t)^2 \quad \text{and} \quad \int_0^t l(\bar{t})^2 d\bar{t} \leq b \cdot l(t)^2 \quad \forall t \in [0, b],$$

the latter as  $l$  is nondecreasing. Using these inequalities and integrating in (14):

$$u(t) - A_1 \int_0^t u \leq A'_0 - 2V(\gamma(t), t) + A_2 b l(t)^2 < C_0 + C_1 l(t)^2,$$

where  $A'_0, C_0, C_1$  are constants ( $C_0$  and  $C_1$  positive), obtained by taking into account the hypothesis (iii). So, putting  $v(t) = \int_0^t u$  and relabelling  $A_1$ ,

$$\dot{v} < C_0 + C_1 \cdot l^2 + C_2 \cdot v \quad \text{for some constants } C_0, C_1, C_2 > 0. \quad (15)$$

Now,  $v$  can be regarded as a subsolution of a differential equation, and Lemma 1 will be applicable to the solution  $v_0$  of this equation with  $v_0(0) = v(0) = 0$  i.e.  $v(t) < v_0(t)$  and, taking into account (15):

$$\dot{v} < C_0 + C_1 \cdot l^2 + C_2 \cdot v_0 = \dot{v}_0$$

on  $(0, b)$ . As  $u = \dot{v}$ , to bound  $\dot{v}_0$  would suffice.

Notice that  $v_0$  can be written explicitly as:

$$v_0(t) = e^{C_2 t} \int_0^t e^{-C_2 \bar{t}} (C_0 + C_1 l(\bar{t})^2) d\bar{t}$$

so that, using that  $l$  is nondecreasing,

$$\dot{v}_0 \leq C_0 + C_1 l^2 + C_2 b e^{C_2 b} (C_0 + C_1 l^2) = A + B l^2 \quad \text{on } [0, b] \quad (16)$$

for some constants  $A, B > 0$ . But recall that  $\dot{l} = \sqrt{u} < \sqrt{\dot{v}_0}$ , that is,  $l$  can be also regarded as a subsolution of a differential equation:

$$\dot{l} < \sqrt{A + B \cdot l^2}. \quad (17)$$

So,  $l$  is bounded by the corresponding solution ( $l(t) < \sqrt{A/B} \cdot \sinh(\sqrt{B} \cdot t)$  on  $(0, b)$ ) and, thus,  $u$  (regarded either as  $\dot{l}^2$  in (17) or as  $\dot{v}_0$  in (16)) is bounded, as required.

*Step 3: As  $g$  is complete,  $\dot{\gamma}$  must lie in a compact subset.* The aim is to prove the extendability of  $\dot{\gamma}$  as an integral curve of the vector field  $G$  on  $T(M \times \mathbb{R})$  defined in the first step. As a first consequence of the boundedness of  $u$ , the completeness of  $g$  imply that  $\gamma$  must be convergent in  $M$ . Then, it is convenient to distinguish two type of reasonings:

(3a) In the case that  $M$  is finite dimensional, the convergence of  $\gamma$  at  $b$ , the boundedness of  $u = g(\dot{\gamma}, \dot{\gamma})$  and the local compactness of  $TM$ , are enough to ensure that  $\dot{\gamma}$  lies in a compact subset of  $TM$ , so that Proposition 1 is applicable to  $G$ .

(3b) In the infinite-dimensional case, the lack of local compactness requires a more elaborated argument. First, the Riemannian metric  $g$  on  $M$  induces naturally a Riemannian metric  $\tilde{g}$  on  $TM$ , the *Sasaki metric* [56]. As proven by Ebin [22],  $\tilde{g}$  is complete whenever so is  $g$ . The vector field  $G$  can be written as a sum  $G = G_0 + G_1 + G_2$  where  $G_0$  is the geodesic spray and, thus, a horizontal vector field,  $G_1$  is a vertical vector field such that, at each  $v_{(p,t)}$ , depends only of the value of  $R + \nabla^M V$  at  $(p, t)$  and  $G_2$  is also a vertical vector which, at each  $v_{(p,t)}$ , can be identified with  $E(v_{(p,t)})$ . The convergence of  $\gamma$  yields a bound for  $\tilde{g}(G_1, G_1)$  on  $\dot{\gamma}$ , the boundedness of  $u$  implies a bound for  $\tilde{g}(G_0, G_0)$  and, then, the boundedness of the operator  $E$  implies the boundedness of  $\tilde{g}(G_2, G_2)$ . So,  $G$  is bounded on  $\dot{\gamma}$ , and Proposition 2 is applicable.

*Remark 5.* (1) The result can be also sharpened, if one is only interested in the forward or backward completeness of the trajectories, i.e. the possibility to extend the solutions to an upper or lower unbounded interval type  $[a, \infty)$  or  $(-\infty, a]$  (also called positive or negative completeness). From the proof is clear that, in order to obtain the extensibility of the trajectories to  $+\infty$  (resp.  $-\infty$ ), one requires only the upper (resp. lower) uniform bound of  $g(v, S(v))/g(v, v)$ , for  $v \in TM \setminus \{0\}$ ,<sup>4</sup> as well as the upper (resp. lower) bound of  $\partial V / \partial t$ , instead of the bounds for the norm and absolute value imposed in the hypotheses (i) and (iii).

(2) As a trivial consequence of Theorem 2, if  $M$  is compact then all the inextendible trajectories are complete, for any  $E, R, V$ .

**Primary and positively complete functions.** The optimal growth allowed either for  $-V$  or for  $|\partial V / \partial t|$  can be sharpened, by using bounds in the spirit of the *primary* ones, introduced for Theorem 1, which are clearly related to the notion of *positive completeness* introduced by Abraham and Marsden [62].

A smooth function  $V_0 : [0, \infty) \rightarrow \mathbb{R}$  is called *positively complete* if it is non-increasing and satisfies

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<sup>4</sup> This can be rephrased as a bound of the spectrum of  $S$ , see [40, Th. 3.10].

$$\int_0^{+\infty} \frac{ds}{\sqrt{e - V_0(s)}} = \infty,$$

for some (and then all) constant  $e > V_0(0)$  (hence  $e > V_0(s)$  for all  $s \in [0, +\infty)$ ). Extending Abraham-Marsden notions, we say that a smooth time-dependent function  $V : M \times \mathbb{R} \rightarrow \mathbb{R}$  is bounded by a positively complete function along finite times if there exists functions  $V_0, C : [0, \infty) \rightarrow \mathbb{R}$ ,  $V_0$  positively complete and  $C > 0$  such that:

$$V(p, t) \geq C(t)V_0(|p|) \quad \forall (p, t) \in M \times \mathbb{R}.$$

The relation between these notions and those used in the last subsection comes from the fact that a smooth function  $V_0$  is positively complete if and only if  $\sqrt{e - V_0}$  is well-defined and primarily complete for some  $e > V_0(0)$ . Now, from the proof of Theorem 2, one can check easily:

Hypotheses (ii) and (iii) in Theorem 2 can be replaced by the following more general one: there exists a primarily complete function  $\alpha$  and a positive one  $C$  such that  $R$  is primarily bounded along finite times by  $C \cdot \alpha$  and  $-V(p, t), |\partial V / \partial t|(p, t) < C(t)^2 \alpha(|p|)^2$  for all  $(p, t) \in M \times \mathbb{R}$ .

In particular, the at most quadratic bounds in (iii) can be improved by requiring only bounds<sup>5</sup> by, say,  $C_0(t) + C_2(t)|x|^2 \log^2(1 + |x|)$  and the at most linear bound in (ii) by  $\tilde{C}_0(t) + C_1(t)|x| \log(1 + |x|)$  (or other functions pointed out in [1, p. 233] or [14, Remark 5(2)]). These bounds might be optimized further, combining them also with better bounds for  $E$ .

**The time-dependence of the potential  $V$ .** For a non-autonomous potential, the role of the bounds of  $\partial V / \partial t$  becomes quite subtler. Notice that one can regard  $\nabla^M V$  as a time-dependent vector. Thus:

If we assume in Theorem 2 that  $\nabla^M V$  grows at most linearly along finite times, no bound for  $\partial V / \partial t$  is necessary. Nevertheless, such a hypothesis is not more general than the stated one (iii). In fact, in the autonomous case, if  $\nabla^M V$  grows at most linearly then  $-V$  grows at most quadratically, but, clearly, the converse does not hold.

Other alternative bounds for  $\partial V / \partial t$  in Theorem 2 can be explored. For example, assuming by simplicity  $R = 0$  in (ii), the result of completeness still holds if we replace (iii) by the following two conditions:  $V$  is lower bounded at finite times ( $V(p, t) \geq -C_0(t)$ ) and:

$$|\partial V / \partial t| \leq C_1(t)(V(p, t) - C_0(t)) \quad \forall (p, t) \in M \times \mathbb{R}. \quad (18)$$

<sup>5</sup> These improvements can be also extended to other contexts, as the completeness of certain Finler metrics in [20].

In fact, (13) would yield now  $d(u + 2V)/dt < C(u + 2V - B)$  for some constants  $C > 0, B \in \mathbb{R}$  which depend on the domain  $[0, b), b < \infty$ . So,  $u + 2V$  (and, then,  $u$ ) would be bounded as a subsolution, see [16] for details).

These new bounds (lower for  $V$  plus (18)) are independent of those in (iii) because, when  $V$  grows fast to infinity, such a growth is allowed for  $\partial V / \partial t$  too. So, to find a general optimal bound for  $\partial V / \partial t$  (say, with some geometric interpretation) remains as a natural question.

### 3.2.2 Notes on the general Finsler case

**Finsler metrics and second order equations.** In order to extend previous results to the Finslerian setting, notice that the Riemannian metric  $g$  in Theorem 2 not only allows to introduce distances and estimates on the growth of tensor fields, but also becomes essential to pose the second-order differential equation (12). For the Finslerian extension, this implies not only higher differentiability for the Finsler metric  $F$  but also its *strong convexity*.

*Remark 6.* As pointed out in Section 2, the existence of smooth Finsler metrics introduce some restrictions in the infinite dimensional case. In fact, notions such as pseudo-gradients<sup>6</sup> were introduced to avoid those restrictions. Recall that the smoothness of each pointwise norm  $F_p$  is required only away from 0 and, thus, it can be characterized as the smoothness of the  $F_p$ -unit sphere as a submanifold of the corresponding vector space  $T_p M$ . However, the smoothness of  $F$  is not enough to introduce connections, covariant derivatives, etc.

The triangle inequality implies that, for each norm  $F_p$ ,  $p \in M$ , the closed unit ball  $\bar{B}_p(0, 1)$  is convex, i.e., it contains any segment with endpoints in  $\bar{B}_p(0, 1)$ . If the triangle inequality holds strictly, then the unit sphere is strictly convex, in the sense that each segment with endpoints in  $\bar{B}_p(0, 1)$  must be entirely contained in the open unit ball  $B_p(0, 1)$  except, at most, the endpoints. Nevertheless, even in the smooth finite-dimensional case, the unit sphere may be strictly convex but not strongly convex. To define this notion, recall that the *fundamental tensor* of each norm  $F_p$  is the tensor field on  $T_p M \setminus \{0\}$  defined as the Hessian  $h_{v_p}$  of  $F_p^2$  at each  $v_p \in T_p M \setminus \{0\}$ . Such a Hessian can be defined by using the affine connection of  $T_p M$  if  $F_p$  is  $C^2$ . Now, consider the slit tangent bundle  $TM \setminus \{0\}$  and the tangent bundle  $TM$ , as well as the natural projection  $\pi : TM \setminus \{0\} \rightarrow M$ . This maps induces a vector bundle  $\pi^*(TM)$  with base  $TM \setminus \{0\}$ , being its fiber at each  $v \in TM \setminus \{0\}$  isomorphic to  $T_{\pi(v)} M$ . Taking the fundamental tensor for each  $F_p$ ,  $p \in M$ , one defines naturally the

<sup>6</sup> According to Palais [46, Defn. 4.1] (and taking into account Moore's modification [43, p. 50]), a pseudo-gradient for a function  $V$  on an open subset  $U$  is a locally Lipschitz vector field  $X$  such that  $\varepsilon^2 F_p(X_p)^2 \leq \|dV_p\| \leq \varepsilon^{-2} dV_p(X_p)$  for all  $p \in U$ .



fundamental tensor field  $h$  of  $F$  as a tensor field on the vector bundle  $\pi^*(TM)$ , and  $F$  is called strongly convex when  $h$  becomes a smooth positive definite tensor.

Strong convexity may introduce a new restriction in the infinite-dimensional case, but it is necessary for several purposes, even in the case of  $n$ -manifolds (see [34] for details):

- To ensure that geodesics (defined as extremals of the energy functional) are determined univocally by its initial condition (starting point and velocity) at some point. That is, otherwise *geodesics cannot be regarded as solutions of a second order differential equation* nor their velocities yield integral curves on a vector field on  $TM$ .
- To ensure (at least in the finite-dimensional case) that the natural Legendre transformation  $TM \rightarrow TM^*, v_p \mapsto g_{v_p}(v_p, \cdot)$  (which generalizes the metric isomorphism of inner spaces, see (4), but may not be linear) becomes a diffeomorphism. Recall that this map is the fiber derivative associated to the Lagrangian  $L = F^2/2$  (see [59, Sect. 3.1], [2, Sect. 3.6]) and, then, the Lagrangian becomes hyperregular. In this case gradients can be defined, and pseudo-gradients are no longer necessary.
- To define natural connections on the Finsler manifold.

**Standard Finsler case.** Taking into account the difficulties pointed out above for the general Finsler case, we restrict now to *standard Finsler manifolds* i.e.,  $n$ -manifolds endowed with a  $C^\infty$ -smooth and strongly convex Finsler metric (as, for example, in [3]). Some similarities with the Riemannian case appear:

- A covariant derivative for vector fields on curves. Thus, the acceleration of these curves can be defined, extending so the notion of  $D\dot{\gamma}/dt$  in the Riemannian case [3, pp. 121-124], [59, Sect. 5.3].
- Non-constant geodesics can be defined as curves with 0 acceleration, they admit a variational characterization and they also determine a (second order equation) vector field  $G$  on the slit tangent bundle  $TM \setminus \{0\}$  so that the integral curves of  $G$  are the curves of velocities of geodesics, [3, Sect. 3.8, 5.3], [59, Sect. 5.1].
- The Finsler metric  $F$  provides the fundamental tensor as well as a natural Sasaki type metric on the slit tangent bundle that makes  $TM \setminus \{0\}$  a Riemannian manifold [3, p. 35].

Of course, important differences with the Riemannian case remain, as Chern/Rundt connection (as well as Cartan, Hashiguchi or Berwald connections) in Finslerian geometry becomes much subtler than the natural Levi-Civita connection for the Riemannian case.

With these elements at hand, one can try to give different Finslerian extensions of Theorem 2. Here, we will consider just the most obvious one, and leave the possibility to obtain more general results for further developments. To avoid working with Finslerian machinery and work with one of the possible connections, notice that, in the case  $R = E = 0$ , formula (12) is the Euler-Lagrange equation for the critical curves of the action:

$$\int_a^b \left( \frac{1}{2} F(\dot{\gamma}(t))^2 - V(\gamma(t), t) \right) dt \quad (19)$$

with fixed points  $\gamma(a), \gamma(b)$ . In Theorem 2,  $F$  is the norm of the Riemannian metric but, obviously, functional (12) makes sense for any Finsler metric and, under some the conditions as above, its Euler-Lagrange equation can be written as in (12).

**Proposition 3.** *Let  $(M, F)$  be a standard Finsler manifold, consider a  $C^1$  time-dependent potential  $V : M \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $-V$  and  $|\partial V / \partial t|$  grows at most quadratically for finite times. Then, any inextensible curve  $\gamma : I \subset \mathbb{R} \rightarrow M$  whose restrictions to compact subintervals are critical points of the action functional (19) is complete.*

*Proof.* Notice first that the problem can be reduced to study the integral curves of a vector field on  $TM$ , because, as the Finsler metric is standard, the Lagrangian  $L = (F^2/2) - V$  becomes regular (in fact, hyper-regular), see for example [2, Th. 3.5.17, 3.8.3]. Then, putting  $u = F(\dot{\gamma})^2$ , one has  $d(u + 2V)/dt = 2\partial V / \partial t$  and formula (14) holds (with  $A_1 = 0$ ), so that the proof follows as in Theorem 2.

*Remark 7.* A different direction in the possible generalizations of Theorem 2, is to allow non-reversible metrics in the standard case, so that  $F(v) \neq F(-v)$  in general. This leads to consider *generalized distances* (i.e., possibly non-symmetric ones) and then, *forward and backward* geodesics and Cauchy completions, as well as many other subtleties (see [24] and references therein). Nevertheless, the general background for completeness would remain in this case. In fact, Proposition 3 can be extended to the non-reversible case. Namely, regarding the hypotheses of completeness for  $F$  in the sense of, say, *forward completeness*, and the generalized distance  $d_F$  to the base point in the ordering  $|p| = d_F(p_0, p)$  (so that the bound for the potential remains formally equal), the technique as well as the conclusion of *forward completeness* still hold.

## 4 Completeness of pseudo-Riemannian geodesics

This section is divided into four parts. The first one tries to orientate the intuition on completeness on indefinite manifolds by recalling some examples and comparing the role of incompleteness in relativistic singularity theorems with bounds on the diameter for Riemannian distances in some Myer's type results. In the second part, we recall some results on completeness for manifolds with a high number of symmetries, being apparent the difference between global symmetries (homogeneous, symmetric spaces) as in Theorems 5, 6 and local ones (constant curvature, local symmetry) in Theorems 7, 8. The third part is focused in plane wave type spacetimes, whose completeness yield a direct link with the Riemannian results of trajectories under potentials, Theorem 9. Previous results suggest some open questions stated in the last part of the section.

In what follows,  $(M, g)$  will be a  $n$ -manifold endowed with a pseudo-Riemannian metric of index  $\nu$ , typically a Lorentzian one (i.e.,  $\nu = 1$  so that the signature is  $(-, +, \dots, +)$ ). The name of *semi-Riemannian* manifold (instead of pseudo-Riemannian) has been also spread, especially since O'Neill's book [44]. This book is referred here for general background on pseudo-Riemannian geometry, the review [17] for the specific problem of geodesic completeness, and the book [5] for related Lorentzian results.

#### 4.1 The pseudo-Riemannian and Lorentzian settings

Let  $(M, g)$  be a pseudo-Riemannian manifold, and  $v \in TM$ ,  $v \neq 0$ . Extending the nomenclature in General Relativity,  $v$  will be called *timelike* (resp. *lightlike*, *spacelike*) if  $g(v, v) < 0$  (resp.  $= 0$ ,  $> 0$ ).

**Abandoning Riemannian intuition.** For a pseudo-Riemannian manifold there is no any result analogous to Hopf-Rinow one and, for example,  $M$  may be compact and geodesically incomplete.

*Example 1.* Consider the Lorentzian metric  $g$  on  $\mathbb{R}^2$  defined as  $g = 2dxdy + \tau(x)dy^2$ , where  $\tau$  is periodic of period 1,  $\tau(0) = 0$  and  $\tau'(0) \neq 0$ . A simple computation shows that the line  $x = 0$  can be reparameterized as an incomplete lightlike geodesic. So, the quotient torus  $T = \mathbb{R}^2 / \mathbb{Z}^2$  inherits an incomplete Lorentzian metric (more refined properties on tori can be found in [55] and referencies therein).

The previous example also shows that a closed lightlike geodesic may be non-periodic and, then, incomplete. Also as a difference with the Riemannian case, a homogeneous Lorentzian manifold may be incomplete.

*Example 2.* Consider a half plane of Lorentz-Minkowski space in lightlike coordinates  $u, v$  namely  $(\mathbb{R}^+ \times \mathbb{R}, g = 2dudv)$ . This space is trivially incomplete, and it is homogeneous too, as both, the  $v$ -translations and the maps  $\Phi_\lambda : (u, v) \mapsto (\lambda u, v/\lambda)$  (for any  $\lambda > 0$ ), are isometries. Recall also that the quotient cylinder obtained from the orbits of the isometry group  $\{\Phi_2^m : m \in \mathbb{Z}\}$  is another example of space with a closed incomplete lightlike geodesic (namely, the projection of  $u \mapsto (u, 0)$ ).

**Singularity theorems.** Even though at the very beginning of General Relativity incompleteness was regarded as a pathological property for a physical spacetime, the further development of Relativity showed that incompleteness appears commonly under physical conditions. Well-known results in this direction were obtained by Raychaudhuri [50], Penrose [49], Hawking [30], Gannon [27] or, more recently, Galloway and Senovilla [26], among others (see for example the review [57]). We emphasize that the claimed incompleteness here occurs only for geodesics of timelike or lightlike type<sup>7</sup>. Even though it is not totally clear at what extent such incomplete geodesics would represent a physical singularity (as well as the meaning of the

<sup>7</sup> Explicit examples by Kundt [33], Geroch [28, p. 531] and Beem [4] showed the full logical independence among spacelike, timelike and lightlike geodesic completeness.

latter, see the classical discussion [28]), the moral in Relativity is that the knowledge of the possible completeness or incompleteness of the underlying Lorentzian manifold becomes an essential property of the spacetime.

As pointed out in [53], perhaps the simplest singularity theorem for researchers interested in connections with Riemannian Geometry is the following one by Hawking, which can be regarded as a support for the physical existence of a *Big Bang*.

**Theorem 3.** *Let  $(M, g)$  be a spacetime satisfying the following conditions:*

1.  $(M, g)$  is globally hyperbolic,
2. *there exists some spacelike Cauchy hypersurface  $S$  with an infimum  $C > 0$  of its expansion, that is, such that its mean curvature vector  $\mathbf{H} = H\mathbf{n}$ , where  $\mathbf{n}$  is the future-directed unit normal, satisfies  $H \geq C > 0$ ,*
3. *the timelike convergence condition holds:  $\text{Ric}(v, v) \geq 0$  for any timelike vector  $v$ .*

*Then, any past-directed timelike curve starting at  $S$  has length at most  $1/C$ .*

The reason is that the proof of this theorem can be regarded as isomorphic to the proof of the following purely Riemannian result:

**Theorem 4.** *Let  $(M, g)$  be a Riemannian manifold satisfying:*

1.  *$g$  is complete,*
2. *there exists some embedded hypersurface  $S$  which separates  $M$  as a disjoint union  $M = M_- \cup S \cup M_+$ , with an infimum  $C > 0$  of its expansion towards  $M_+$ , that is, such that its mean curvature vector  $\mathbf{H} = H\mathbf{n}$ , where  $\mathbf{n}$  is the unit normal which points out  $M_-$ , satisfies  $H \geq C > 0$ ,*
3.  *$\text{Ric}(v, v) \geq 0$  for every  $v$ .*

*Then,  $\text{dist}(p, S) \leq 1/C$  for every  $p \in M_-$ .*

In fact, this last theorem can be proven by using standard techniques on focal points and Myers' theorem. Such techniques can be extended to the Lorentzian setting by realizing that the roles of each one of the three hypotheses in Theorem 3 is isomorphic in the proof to the corresponding hypothesis in Theorem 4 (in particular, the role of Riemannian completeness is played by global hyperbolicity), see [35] for full details. The techniques of singularity theorems, however, become much more refined, because of the weakening of causality assumptions or the appearance of genuinely Lorentzian elements such as trapped surfaces, see for example [31] or [26].

## 4.2 Completeness under symmetries

After previous considerations, it is clear that some strong assumptions will be required in order to prove geodesic completeness. We will focus on some type of symmetries.

**Killing and conformal fields.** The simple Examples 1, 2 of non-complete compact or homogeneous Lorentzian manifolds, make apparent the importance of the following theorem by Marsden [41] (see also [1, 4.2.22]):

**Theorem 5.** [41] *Any compact homogeneous pseudo-Riemannian manifold is geodesically complete.*

Marsden's proof is carried out by proving that  $TM$  can be written as the union of compact subsets  $S_\alpha$ , each one invariant by the geodesic flow (and, so, Proposition 1 yields directly the result). In fact, if  $\mathfrak{g}^*$  is the dual of the Lie algebra of the isometry group, and  $P : TM \rightarrow \mathfrak{g}^*$  is the momentum map (i.e.,  $P(v)\xi = g(v, \xi_M)$ , where  $\xi_M$  is the infinitesimal generator of  $\xi \in \mathfrak{g}$ ), then  $S_\alpha = P^{-1}(\alpha)$ , for each  $\alpha \in \mathfrak{g}^*$ .

As proven by Romero and the author [52, 51], this result can be extended in two directions. Firstly, it is not necessary, in order to ensure the completeness of each geodesic  $\gamma$ , that its velocity  $\dot{\gamma}$  remains in a compact subset of  $TM$ . In the spirit of Proposition 2, it is enough if it remains in a compact subset when its domain is restricted to bounded intervals. From such an observation, previous result can be extended to metrics conformal to Marsden's. Secondly, a homogeneous manifold is plenty of Killing vector fields but if, say, a compact Lorentzian manifold admitted just one *timelike* Killing vector field<sup>8</sup>  $K$ , this would be enough. Indeed, as  $g(\dot{\gamma}, K)$  is a constant for any geodesic  $\gamma$ , this (plus the constancy of  $g(\dot{\gamma}, \dot{\gamma})$ ) is sufficient to ensure that  $\dot{\gamma}$  lies in a compact subset. So, from these ideas:

**Theorem 6.** [52, 51] *A compact pseudo-Riemannian manifold  $(M, g)$  of index  $v$  is geodesically complete if one of the following properties hold:*

- $(M, g)$  is (globally) conformal to a homogeneous one, or
- $(M, g)$  admits  $v$  conformal vector fields which are pointwise independent.

The techniques also admits extensions to non-compact manifolds, see [51], [54]; for applications to classification of spaceforms, see [32]. Further results on locally homogeneous 3-spaces can be found in [9], [18] and [21].

**Locally symmetric and constant curvature manifolds.** As a difference with homogeneous spaces, it is easy to check that any *semi-Riemannian symmetric space* is *geodesically complete* (see for example [44, Lemma 8.20]). Nevertheless, even for locally symmetric spaces and, in particular, constant curvature ones, the problem is not as trivial as it may seem. We quote two results which will be relevant in order to state some open questions below. The first one is due to Lafuente:

**Theorem 7.** [39] *For a locally symmetric Lorentzian manifold, the three types of causal completeness (timelike, lighlike and spacelike) are equivalent.*

The second one was proven by Carrière [18] in the flat case and extended by Klinger [37] for manifolds of any constant curvature.

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<sup>8</sup> This case is interesting also for the classification of flat compact Lorentzian manifolds, which are called then *standard*, see [32].

**Theorem 8.** [18, 37] *Any compact Lorentzian manifold of constant curvature is geodesically complete.*

*Remark 8.* Recall that the proof of this result holds only for Lorentzian signature; as far as we know, the extension of the result to higher signatures is an open problem.

### 4.3 Riemannian and Lorentzian interplay: planes waves

**Plane waves, pp-waves and further generalizations.** Consider a Lorentzian  $n$ -manifold,  $n \geq 3$ , that can be written globally as  $(M = \mathbb{R}^2 \times M_0, g)$  where the natural coordinates of  $\mathbb{R}^2$  will be labelled  $(u, v)$  and  $g$  is written as:

$$g_{(u,v,x)} = -2dudv + H(u, x)du^2 + \Pi_0^* g_0, \quad \forall (u, v, x) \in \mathbb{R}^2 \times M_0,$$

being  $\Pi_0 : M \rightarrow M_0$  the natural projection and  $g_0$  a Riemannian metric on  $M_0$ , see [13]. Here, we will refer to these spaces as  $M_0$ p-waves. When  $(M_0, g_0)$  is just  $\mathbb{R}^{n-2}$ , these metrics are called *pp-waves* (*plane-fronted waves with parallel rays*), namely,  $M = \mathbb{R}^n$ ,

$$g_{(u,v,x)} = -2dudv + H(u, x^1, \dots, x^{n-2})du^2 + \sum_{i=1}^{n-2} (dx^i)^2 \quad \forall (u, v, x^1, \dots, x^{n-2}) \in \mathbb{R}^n.$$

Such a pp-wave is called a *plane wave* when  $H$  is quadratic in  $(x^1, \dots, x^{n-2})$ ,

$$H(u, x^1, \dots, x^{n-2}) = \sum_{i,j=1}^{n-2} A_{ij}(u) x^i x^j.$$

In the particular case  $n = 4$  one writes  $H(u, x, y) = a(u)(x^2 - y^2) + 2b(u)xy + c(u)(x^2 + y^2)$ , where  $a, b, c$  are arbitrary smooth functions of  $u$ . The functions  $a, b$  describe the wave profiles of the two linearly independent polarization modes of gravitational radiation, while  $c$  describes the wave profile of non-gravitational radiation. When  $c = 0$  (vacuum or gravitational plane waves) the Ricci tensor vanishes.

Plane waves are interesting in many physical issues. We remark here that they are also interesting in the framework of *rth-symmetric spaces*  $r \geq 2$  (introduced in [58], see [7] for a systematic study). These are pseudo-Riemannian manifolds with  $r$ th-covariant derivative of its curvature tensor  $R$  equal to 0:

$$\nabla^r R := \nabla \dots^{(r)} \nabla R \equiv 0.$$

For Riemannian manifolds  $r$ th-symmetry implies local symmetry (i.e.,  $\nabla R = 0$ ) but proper examples of  $r$ th-symmetric spaces can be found in the class of plane waves. In fact, such examples are obtained just regarding the matrix  $A$  as a polynomial in  $u$  of degree  $r - 1$ :

$$A_{ij}(u) = a_{ij}^{(r-1)} u^{r-1} + \dots + a_{ij}^{(1)} u^1 + a_{ij}^{(0)}$$

where  $a_{ij}^{(r-1)} \not\equiv 0$ ; a simple computation shows that  $\nabla^r R = 0$  but  $\nabla^{r-1} R \neq 0$ .

As shown in [7], proper 2nd-symmetric Lorentzian spaces are locally isometric to the product of such a wave (with  $r = 2$ ) and a locally symmetric Riemannian space.

**Completeness of  $M_0$ p-waves.** A nice relation between the geodesic completeness of a class of Lorentzian manifold and the completeness of Riemannian trajectories for a potential appears in the case of  $M_0$ p-waves:

**Theorem 9.** *A  $M_0$ p-wave is geodesically complete if and only if  $(M_0, g_0)$  is complete and the trajectories of*

$$\frac{D\dot{\gamma}}{dt}(t) = -\nabla^{M_0} V(\gamma(t), t)$$

*are complete for  $V = -H/2$ .*

*Thus, under completeness of the Riemannian part  $(M_0, g_0)$ , a  $M_0$ p-wave is complete if  $H$  and  $|\partial H / \partial u|$  grows at most quadratically for finite  $u$ -times. In particular, all plane waves are geodesically complete.*

*Proof.* The first part is proven in [13, Th. 3.2], by means of a careful equivalence between the Lorentzian geodesics and Riemannian trajectories [13, Prop. 3.1]. So, it is enough to apply theorem 2.

As emphasized in [14], this type of result also justifies that all physically reasonable pp-waves (that is, those with a qualitative behavior of  $H$  as a plane wave, eventually with a possible decay at infinity) will be geodesically complete and, so, they can be regarded as singularity free.

#### 4.4 Some open questions

Taking into account previous considerations, the following questions become natural and are open, as far as we know:

1. Assume that a compact Lorentzian manifold is globally conformal to a manifold of constant curvature. Must it be geodesically complete?

Recall that this poses a possible extension of Theorem 8, which may be expected after the conformal extension in Theorem 6 of Marsden's Theorem 5. It is also worth pointing out that, for compact manifolds, *lightlike* completeness is a conformal invariant (this is easy to check as lightlike pregeodesics are conformally invariant, and their reparameterizations as geodesics depend on a bounded conformal factor, see [17, Section 2.3] for detailed computations). So, if a counterexample to the question existed, it would be incomplete in some causal sense and complete in the lightlike case. In particular, this would prove that Lafuente's Theorem 7 cannot be extended to the conformal case even for compact manifolds.

2. Assume that a pseudo-Riemannian manifold is  $r$ -th symmetric. Must the three types of causal completeness be equivalent?

Such a question becomes natural after Lafuente's Theorem 7, especially in the case of Lorentzian 2nd-symmetric spaces, because of their simple classification explained above.

3. Must any complete gravitational (i.e., Ricci flat) pp-wave be a plane wave?

This is a long-standing open problem posed by Ehlers and Kundt [23]. Recall first that all plane waves are complete, even if non-gravitational (Theorem 9). The fact that these waves are gravitational, i.e., Ricci flat, yields a link with complex variable, as this condition is equivalent to the harmonicity of  $H(x, u)$  with respect to the variable  $x$  (see [25]). Thus, there is both, physical and mathematical motivations for its study [6, 25]. Recall that the completeness of holomorphic vector fields become a field of research in its own right which has been studied with specific tools, see for example [10], [11].

As a last comment, we point out that the completeness of trajectories in a Lorentzian manifold under external forces is an almost open field with rich possibilities [15]. So, even though the physical interpretations of such forces are less apparent in the Lorentzian case than in the Riemannian one, this may be an interesting topic for future research.

## References

1. Abraham R., Marsden, J.E.: *Foundations of Mechanics*, 2nd Ed. Addison-Wesley Publishing Co., Boston (1987).
2. Abraham R., Marsden J.E., Ratiu T.: *Manifolds, Tensor Analysis and Applications*, 2nd Ed., Springer, New York (1988).
3. Bao, D., Chern, S.-S., Shen, Z.: *An introduction to Riemann-Finsler geometry*. Graduate Texts in Mathematics, 200. Springer-Verlag, New York (2000).
4. Beem, J.K.: Some examples of incomplete space-times. *General Relativity Gravit.* **7** 501-509 (1976).
5. Beem, J.K., Ehrlich, P.E., Easley, K.L.: *Global Lorentzian Geometry*, Monographs Textbooks Pure Appl. Math. 202, Dekker Inc., New York (1996).
6. Bicak, J.: The role of exact solutions of Einsteins equations in the developments of general relativity and astrophysics selected themes. *Lect. Notes Phys.* **540** 1-126 (2000).
7. Blanco, O.F., Sánchez, M., Senovilla, J.M.M.: Structure of second-order symmetric Lorentzian manifolds *J. Eur. Math. Soc.* **15** 595-634 (2013).
8. Bonic, R., Frampton, J.: Smooth functions on Banach manifolds. *J. Math. Mech.* **15** 877-898 (1966).
9. Bromberg, S., Medina, A.: Geodesically Complete Lorentzian Metrics on Some Homogeneous 3 Manifolds. *SIGMA Symmetry, Integrability and Geometry: Methods and Applications* 4 088, 13pp (2008).



10. Brunella, M.: Complete polynomial vector fields on the complex plane. *Topology* 43, no. 2, 433-445 (2004).
11. Bustinduy, A., Giraldo, L.: Completeness is determined by any non-algebraic trajectory. *Adv. Math.* 231 no. 2, 664-679 (2012).
12. Calvaruso, G.: Homogeneous structures on three-dimensional Lorentzian manifolds. *J. Geom. Phys.* 57, no. 4, 1279-1291 (2007).
13. Candela, A., Flores, J.L., Sánchez, M.: On general plane fronted waves. *Geodesics. Gen. Relativ. Gravit.* 35, 631-649 (2003)
14. Candela, A.M., Romero A., Sánchez, M.: Completeness of the trajectories of particles coupled to a general force field. *Arch. Ration. Mech. Anal.*, **208** Issue 1 255-274 (2013).
15. Candela, A.M., Romero A., Sánchez, M.: Completeness of relativistic particles under stationary magnetic fields. In: *Proc. XXI International Fall Workshop on Geometry and Physics, International Journal of Geometric Methods in Modern Physics* (to appear).
16. Candela, A.M., Romero A., Sánchez, M.: Remarks on the completeness of plane waves and the trajectories of accelerated particles in Riemannian manifolds, in: *Proc. Int. Meeting on Differential Geometry* (University of Córdoba, Córdoba, pp. 27-38 (2012).
17. Candela, A.M, Sánchez, M.: Geodesics in semiRiemannian manifolds: geometric properties and variational tools. In: *Recent Developments in pseudoRiemannian Geometry* (D.V. Alekseevsky & H. Baum Eds), Special Volume in the ESI Series on Mathematics and Physics, EMS Publ. House, Zürich, 359-418 (2008).
18. Carrière, Y.: Autour de la conjecture de L. Markus sur les variétés affines. *Invent. Math.* **95** 615-628 (1989).
19. Deimling, K.: *Nonlinear functional analysis*. Springer-Verlag, Berlin (1985).
20. Dirmeier, A., Plaue, M., Scherfner, M.: Growth conditions, Riemannian completeness and Lorentzian causality. *J. Geom. Phys.* **62** no. 3, 604612 (2012); Erratum ibidem (2013).
21. Dumitrescu, S., Zeghib, A.: Géométries lorentziennes de dimension 3: classification et complétude. *Geom. Dedicata* 149, 243-273 (2010).
22. Ebin, D. G.: Completeness of Hamiltonian vector fields. *Proc. Amer. Math. Soc.* 26 632-634 (1970).
23. Ehlers, J., Kundt, K.: *Exact Solutions of the Gravitational Field Equations*. In: *Gravitation: an introduction to current research*, ed. L. Witten, J. Wiley & Sons, New York, (1962).
24. Flores, J.L., Herrera, J., Sánchez M.: Gromov, Cauchy and causal boundaries for Riemannian, Finslerian and Lorentzian manifolds. *Memoirs Amer. Math. Soc.* (to appear), arxiv: 1011.1154.
25. Flores, J.L., Sánchez M.: On the Geometry of pp-Wave Type Spacetimes. *Lect. Notes Phys.* **692**, 79-98 (2006)
26. Galloway, G. J., Senovilla J. M. M.: Singularity theorems based on trapped submanifolds of arbitrary co-dimension. *Classical Quantum Gravity* 27 no. 15, 152002, 10 pp. (2010).
27. Gannon, D.: Singularities in nonsimply connected space-times. *J. Mathematical Phys.* 16, no. 12, 23642367 (1975)
28. Geroch R.P.: What is a singularity in General Relativity, *Ann. Phys. (NY)* 48 526-540 (1970).
29. W.B. Gordon: On the completeness of Hamiltonian vector fields. *Proc. Amer. Math. Soc.* 26 329-331 (1970).
30. Hawking, S.W.: The Occurrence of Singularities in Cosmology. III. Causality and singularities. *Proc. Roy. Soc. Lond. A* 300, 187-201 (1967).
31. Hawking, S.W., Penrose, R.: The Singularities of Gravitational Collapse and Cosmology. *Proc. Roy. Soc. Lond. A* 314 529548 (1970).
32. Kamishima, Y.: Completeness of Lorentz manifolds of constant curvature admitting Killing vector fields. *J. Differential Geom.* 37 , no. 3, 569601 (1993).
33. Kundt, W.: Note on the completeness of spacetimes. *Z. Physik* 172 488489 (1963).
34. Javaloyes, M.A., Sánchez M.: On the definition and examples of Finsler metrics. *Ann. Sc. Norm. Sup. Pisa, Cl. Sci.*, to appear (arxiv: 1111.5066).
35. Javaloyes, M.A., Sánchez M.: *An Introduction to Lorentzian Geometry and its Applications*. Sao Carlos, Rima (2010) ISBN: 978-85-7656-180-4.

36. Klingenberg, W.: Riemannian geometry. de Gruyter Studies in Mathematics, 1. Walter de Gruyter & Co., Berlin-New York (1982).
37. Klingler B.: Complétude des variétés lorentziennes à courbure constante. *Math. Ann.* 306 353-370 (1996).
38. Kriegel A., Michor P.W.: The Convenient Setting of Global Analysis. *Mathematical Surveys and Monographs*, Volume 53, American Mathematical Society, Providence (1997).
39. Lafuente López, J.: A geodesic completeness theorem for locally symmetric Lorentz manifolds. *Rev. Mat. Univ. Complut. Madrid* 1 101110 (1988).
40. Lang, S.: Differential and Riemannian manifolds. Third edition. *Graduate Texts in Mathematics*, 160. Springer-Verlag, New York, 1995.
41. Marsden J.E.: On completeness of homogeneous pseudo-Riemannian manifolds. *Indiana Univ. J.* 22 1065-1066 (1972/73).
42. Margalef Roig, J.; Outerelo Domínguez, E.: Una variedad diferenciable de dimension infinita, separada y no regular, *Rev. Matem. Hispanoamericana* 42 (1982), 51-55.
43. Moore, J. D.: Introduction to Global Analysis, University of California Santa Barbara, CA (2010). Available at [www.math.ucsb.edu/~moore/globalanalysisshort.pdf](http://www.math.ucsb.edu/~moore/globalanalysisshort.pdf)
44. O'Neill, B: Semi-Riemannian Geometry with Applications to Relativity. *Pure Appl. Math.* 103 Academic Press Inc., New York, (1983).
45. Palais, R. S: Morse theory on Hilbert manifolds. *Topology* 2 299-340 (1963).
46. Palais, R. S.: Lusternik-Schnirelman theory on Banach manifolds. *Topology* 5 115-132 (1966).
47. Palais, R. S. Critical point theory and the minimax principle. *Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif, 1968)* Amer. Math. Soc., Providence, 185-212 R.I. (1970).
48. Palais, R. S.: When proper maps are closed. *Proc. Amer. Math. Soc.* 24 835-836 (1970).
49. Penrose, R.: Gravitational collapse and space-time singularities. *Phys. Rev. Lett.* 14 5759 (1965).
50. Raychadhuri Raychadhuri, A. K.: Relativistic cosmology I. *Phys. Rev.* 98, 1123-1126 (1955).
51. Romero A, Sánchez M.: On completeness of certain families of semi-Riemannian manifolds. *Geom. Dedicata* 53 , no. 1, 103117 (1994).
52. A. Romero and M. Sánchez: Completeness of compact Lorentz manifolds admitting a time-like conformal Killing vector field. *Proc. Amer. Math. Soc.* 123, no. 9, 2831-2833 (1995).
53. Sánchez, M.: Cauchy Hypersurfaces and Global Lorentzian Geometry. *Publ. RSME*, 8 143-163 (2006).
54. Sánchez, M.: On the geometry of generalized Robertson-Walker spacetimes: geodesics. *Gen. Relativity Gravitation* 30, no. 6, 915932 (1998).
55. Sánchez, M.: Structure of Lorentzian tori with a Killing vector field. *Trans. Amer. Math. Soc.* 349, no. 3, 10631080 (1997).
56. Sasaki, S.: On the Differential Geometry of Tangent Bundles of Riemannian Manifolds, *Tôhoku Math. J.*, 10 338-354 (1958).
57. Senovilla, José M. M.: Singularity Theorems and Their Consequences. *Gen. Relat. Grav.* 29, No. 5 701-848 (1997).
58. Senovilla José, M. M.: Second-order symmetric Lorentzian manifolds. I. Characterization and general results. *Classical Quantum Gravity* 25 no. 24, 245011, 25 pp. (2008).
59. Shen, Z: Lectures on Finsler geometry. World Scientific Publishing Co., Singapore (2001).
60. Teschl, G.: Ordinary differential equations and dynamical systems. *Grad. Stud. Math.*, Vol. 140. Amer. Math. Soc., Providence (2012).
61. Warner, F.W.: The Conjugate Locus of a Riemannian Manifold. *American Journal of Mathematics*, 87, No. 3, 575-604 (1965).
62. Weinstein A., Marsden J.E.: A comparison theorem for Hamiltonian vector fields. *Proc. Amer. Math. Soc.* 26 629-631 (1970).